

**Original citation:**

Baker, Simon and Steiner, Wolfgang . (2017) On the regularity of the generalised golden ratio function. Bulletin of the London Mathematical Society, 49 (1). pp. 58-70.

**Permanent WRAP URL:**

<http://wrap.warwick.ac.uk/81650>

**Copyright and reuse:**

The Warwick Research Archive Portal (WRAP) makes this work by researchers of the University of Warwick available open access under the following conditions. Copyright © and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable the material made available in WRAP has been checked for eligibility before being made available.

Copies of full items can be used for personal research or study, educational, or not-for profit purposes without prior permission or charge. Provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

**Publisher's statement:**

This is the peer reviewed version of the following article: Baker, S. and Steiner, W. (2017), On the regularity of the generalised golden ratio function. Bull. London Math. Soc., 49: 58–70. doi:10.1112/blms.12002, which has been published in final form at <http://dx.doi.org/10.1112/blms.12002> . This article may be used for non-commercial purposes in accordance with [Wiley Terms and Conditions for Self-Archiving](#).

**A note on versions:**

The version presented here may differ from the published version or, version of record, if you wish to cite this item you are advised to consult the publisher's version. Please see the 'permanent WRAP url' above for details on accessing the published version and note that access may require a subscription.

For more information, please contact the WRAP Team at: [wrap@warwick.ac.uk](mailto:wrap@warwick.ac.uk)

# On the regularity of the generalised golden ratio function

Simon Baker<sup>1</sup> and Wolfgang Steiner<sup>2</sup>

<sup>1</sup>*Mathematics Institute,  
University of Warwick,  
Coventry, CV4 7AL, UK*

<sup>2</sup>*IRIF, CNRS UMR 8243,  
Université Paris Diderot - Paris 7,  
Case 7014, 75205 Paris Cedex 13, France*

*Email contacts:*

`simonbaker412@gmail.com`  
`steiner@liafa.univ-paris-diderot.fr`

September 5, 2016

## Abstract

Given a finite set of real numbers  $A$ , the generalised golden ratio is the unique real number  $\mathcal{G}(A) > 1$  for which we only have trivial unique expansions in smaller bases, and have non-trivial unique expansions in larger bases. We show that  $\mathcal{G}(A)$  varies continuously with the alphabet  $A$  (of fixed size). What is more, we demonstrate that as we vary a single parameter  $m$  within  $A$ , the generalised golden ratio function may behave like  $m^{1/h}$  for any positive integer  $h$ . These results follow from a detailed study of  $\mathcal{G}(A)$  for ternary alphabets, building upon the work of Komornik, Lai, and Pedicini (2011). We give a new proof of their main result, that is we explicitly calculate the function  $\mathcal{G}(\{0, 1, m\})$ . (For a ternary alphabet, it may be assumed without loss of generality that  $A = \{0, 1, m\}$  with  $m \in (1, 2]$ .) We also study the set of  $m \in (1, 2]$  for which  $\mathcal{G}(\{0, 1, m\}) = 1 + \sqrt{m}$ , we prove that this set is uncountable and has Hausdorff dimension 0. We show that the function mapping  $m$  to  $\mathcal{G}(\{0, 1, m\})$  is of bounded variation yet has unbounded derivative. Finally, we show that it is possible to have unique expansions as well as points with precisely two expansions at the generalised golden ratio.

*Mathematics Subject Classification* 2010: 11A63, 28A80.

*Key words and phrases:* Beta expansions, Generalised golden ratios, Combinatorics on words

## 1 Introduction and statement of results

Let  $A := \{a_0, a_1, \dots, a_d\}$  be a set of real numbers satisfying  $a_0 < a_1 < \dots < a_d$ . We call  $A$  an alphabet. Given  $\beta > 1$  and  $x \in \mathbb{R}$ , we say that a sequence  $(u_k)_{k=1}^\infty \in A^\mathbb{N}$  is a  $\beta$ -expansion for  $x$  over the alphabet  $A$  if

$$x = \sum_{k=1}^{\infty} \frac{u_k}{\beta^k}.$$

When the underlying alphabet is obvious we may simply refer to  $(u_k)$  as a  $\beta$ -expansion. Expansions in non-integer bases were introduced by Rényi [8]. Perhaps the most well studied case is when  $\beta \in (1, 2]$  and  $A = \{0, 1\}$ . For  $\beta \in (1, 2]$  and this choice of alphabet,  $x$  has a  $\beta$ -expansion over  $A$  if and only if  $x \in [0, \frac{1}{\beta-1}]$ . Moreover, a result of Erdős, Joó, and Komornik [3] states that if  $\beta \in (1, \frac{1+\sqrt{5}}{2})$  then every  $x \in (0, \frac{1}{\beta-1})$  has a continuum of  $\beta$ -expansions. This result is complemented by a theorem of Daróczy and Katai [2] which states that if  $\beta \in (\frac{1+\sqrt{5}}{2}, 2]$  then there exists  $x \in (0, \frac{1}{\beta-1})$  with a unique  $\beta$ -expansion. Note that the end points of the interval  $[0, \frac{1}{\beta-1}]$  trivially have a unique  $\beta$ -expansion for any  $\beta \in (1, 2]$ . The

above demonstrates that the golden ratio acts as a natural boundary between the possible cardinalities the set of expansions can take. It is natural to ask whether such a boundary exists for more general alphabets.

Before we state the definition of a generalised golden ratio it is necessary to define the univoque set. Given an alphabet  $A$  and  $\beta > 1$  we set

$$\mathcal{U}_\beta(A) := \left\{ (u_k)_{k=1}^\infty \in A^\mathbb{N} : \sum_{k=1}^\infty \frac{u_k}{\beta^k} \text{ has a unique } \beta\text{-expansion} \right\}.$$

We call  $\mathcal{U}_\beta(A)$  the univoque set. Note that for any  $\beta > 1$  and alphabet  $A = \{a_0, \dots, a_d\}$  satisfying  $a_0 < a_1 < \dots < a_d$ , the points

$$\sum_{k=1}^\infty \frac{a_0}{\beta^k} \text{ and } \sum_{k=1}^\infty \frac{a_d}{\beta^k}$$

both have a unique expansion, so  $\overline{a_0}$  and  $\overline{a_d}$  are always contained in the univoque set. Here and throughout  $\overline{w}$  denotes the infinite periodic word with period  $w$ . We are now in a position to define a generalised golden ratio for an arbitrary alphabet. Given an alphabet  $A$ , we call  $\mathcal{G}(A) \in (1, \infty)$  the generalised golden ratio for  $A$  if whenever  $\beta \in (1, \mathcal{G}(A))$  we have  $\mathcal{U}_\beta(A) = \{\overline{a_0}, \overline{a_d}\}$ , and if  $\beta > \mathcal{G}(A)$  then  $\mathcal{U}_\beta(A)$  contains a non-trivial element.

Komornik, Lai, and Pedicini [4] were the first authors to make a thorough study of generalised golden ratios over arbitrary alphabets. Importantly they proved that for any alphabet  $A$  a generalised golden ratio exists. For ternary alphabets, they showed that the generalised golden ratio varies continuously with the alphabet. We extend this result to alphabets of arbitrary size.

**Theorem 1.** *Let  $\Delta_d := \{(a_0, a_1, \dots, a_d) \in \mathbb{R}^{d+1} : a_0 < a_1 < \dots < a_d\}$ ,  $d \geq 1$ . The map  $(a_0, a_1, \dots, a_d) \mapsto \mathcal{G}(\{a_0, a_1, \dots, a_d\})$  is continuous on  $\Delta_d$ .*

We prove this theorem in Section 2. In the rest of the paper, we restrict our attention to ternary alphabets. Every ternary alphabet can be assumed to be of the form  $A = \{0, 1, m\}$  for some  $m > 1$  because shifting the alphabet and multiplying by a constant does not affect the generalised golden ratio. We thus set

$$\mathcal{G}(m) := \mathcal{G}(\{0, 1, m\}) \text{ and } \mathcal{U}_\beta(m) := \mathcal{U}_\beta(\{0, 1, m\}).$$

Moreover,

$$\mathcal{G}(m) = \mathcal{G}(\{0, 1, m\}) = \mathcal{G}(\{-m, 1 - m, 0\}) = \mathcal{G}\left(\left\{\frac{m}{m-1}, 1, 0\right\}\right) = \mathcal{G}\left(\frac{m}{m-1}\right).$$

By the above  $\mathcal{G}(m) = \mathcal{G}(\frac{m}{m-1})$  and we may therefore assume  $m \in (1, 2]$ . The authors of [4] considered  $m \geq 2$ , and their results read as follows in our setting.

**Theorem KLP.** *The function  $\mathcal{G} : (1, 2] \rightarrow \mathbb{R}$  is continuous and satisfies*

$$2 \leq \mathcal{G}(m) \leq 1 + \sqrt{m}$$

*for all  $m \in (1, 2]$ . Moreover, the following statements hold.*

- $\mathcal{G}(m) = 2$  for  $m \in (1, 2]$  if and only if  $m = \frac{2^k}{2^k - 1}$  for some positive integer  $k$ .
- The set  $\mathfrak{M} := \{m \in (1, 2] : \mathcal{G}(m) = 1 + \sqrt{m}\}$  is a Cantor set, its largest element is  $x^2 \approx 1.7548$  where  $x \approx 1.3247$  is the smallest Pisot number.
- Each connected component  $(m_1, m_2)$  of  $(1, x^2) \setminus \mathfrak{M}$  has a point  $\mu$  such that  $\mathcal{G}$  is strictly decreasing on  $[m_1, \mu]$  and strictly increasing on  $[\mu, m_2]$ ;  $\mathcal{G}$  is strictly increasing on  $[x^2, 2]$ .

In Section 3, we reprove all these results, making some of the statements more explicit and simplifying several proofs. An approximation of the graph of  $\mathcal{G}$  can be found in Figure 3.1. The function  $\mathcal{G}$  is given by implicit equations on subintervals of  $(1, 2]$ , and it has the following unusual regularity properties.

**Theorem 2.** *The function  $\mathcal{G} : (1, 2] \rightarrow \mathbb{R}$  is differentiable except on the set  $\mathfrak{M}$  and on the countable set of points  $\mu$  defined in Theorem KLP. Its derivative is unbounded, but its total variation is less than 2.*

We have the following result on the size of  $\mathfrak{M}$ .

**Theorem 3.** *The set  $\mathfrak{M}$  is an uncountable Cantor set with Hausdorff dimension 0.*

On certain intervals, the function  $\mathcal{G}$  has the following simple form.

**Theorem 4.** *Let  $h$  be a positive integer and  $2^h \leq m \leq (1 + \sqrt{\frac{m}{m-1}})^h$ . Then we have*

$$\mathcal{G}(m) = \mathcal{G}\left(\frac{m}{m-1}\right) = m^{1/h}.$$

Note that if  $m = 2^h$  then  $m < (1 + \sqrt{\frac{m}{m-1}})^h$ , thus the set of  $m$  defined within Theorem 4 is non-empty.

Moreover, we have the following result on the size of the set of expansions at the generalised golden ratio.

**Theorem 5.** *There exists  $m \in (1, 2]$  such that:*

- $\mathcal{U}_{\mathcal{G}(m)}(m)$  contains non-trivial elements.
- When  $\beta = \mathcal{G}(m)$  there exists an  $x$  with precisely two  $\beta$ -expansions.

Finally, we remark that the problem of calculating  $\mathcal{G}(A)$  remains wide open. Only  $\mathcal{G}(\{0, 1, \dots, m\})$  has been calculated for any positive integer  $m$  in [1].

## 2 Continuity of $\mathcal{G}(A)$

Before proving Theorem 1, we recall results of Pedicini [7, Proposition 2.1 and Theorem 3.1] on (unique) expansions in non-integer bases over arbitrary alphabets; see also [4, Theorem 2.2].

**Theorem P.** *Let  $\beta \in (1, q(A)]$ , with*

$$q(A) := 1 + \frac{a_d - a_0}{\max\{a_1 - a_0, a_2 - a_1, \dots, a_d - a_{d-1}\}}.$$

*Every  $x \in [\frac{a_0}{\beta-1}, \frac{a_d}{\beta-1}]$  has a  $\beta$ -expansion over  $A$ . We have  $u_1 u_2 \dots \in \mathcal{U}_\beta(A)$  if and only if, for all  $i \geq 1$ ,*

$$\sum_{k=0}^{\infty} \frac{u_{i+k}}{\beta^k} < a_{j+1} + \frac{a_0}{\beta-1} \quad \text{when } u_i = a_j \neq a_d, \quad (2.1)$$

*and*

$$\sum_{k=0}^{\infty} \frac{u_{i+k}}{\beta^k} > a_{j-1} + \frac{a_d}{\beta-1} \quad \text{when } u_i = a_j \neq a_0. \quad (2.2)$$

**Remark 2.1.** The conditions (2.1) and (2.2) can be restated in terms of uniqueness regions  $E_a$ : Let

$$E_{a_0} = \left[ \frac{a_0 \beta}{\beta-1}, a_1 + \frac{a_0}{\beta-1} \right), E_{a_j} = \left( a_{j-1} + \frac{a_d}{\beta-1}, a_{j+1} + \frac{a_0}{\beta-1} \right), 1 \leq j < d, E_{a_d} = \left( a_{d-1} + \frac{a_d}{\beta-1}, \frac{a_d \beta}{\beta-1} \right].$$

Then (2.1) and (2.2) hold if and only if  $\sum_{k=0}^{\infty} \frac{u_{i+k}}{\beta^k} \in E_{u_i}$ .

By the following lemma, it is sufficient to consider  $\beta \leq q(A)$ .

**Lemma 2.2.** *We have  $\mathcal{G}(A) \leq q(A)$ .*

*Proof.* If  $\beta > 1 + \frac{a_d - a_0}{a_{j+1} - a_j}$  for some  $0 \leq j < d$ , then  $a_j + \frac{a_d}{\beta-1} < a_{j+1} + \frac{a_0}{\beta-1}$  and thus  $a_j \overline{a_d} \in \mathcal{U}_\beta(A)$ .  $\square$

**Remark 2.3.** This upper bound is attained for certain alphabets. For example, let  $A = \{0, 1, 4, 5\}$ . For  $\beta = q(A) = 8/3$ , the uniqueness regions are  $E_0 = [0, 1)$ ,  $E_1 = (3, 4)$ ,  $E_4 = (4, 5)$  and  $E_5 = (7, 8]$ . If  $\sum_{k=0}^{\infty} \frac{u_{i+k}}{\beta^k} \in E_{u_i}$ , then  $\sum_{k=0}^{\infty} \frac{u_{i+1+k}}{\beta^k} \in (E_{u_i} - u_i)\beta$ ; the latter intervals are  $[0, 8/3)$ ,  $(16/3, 8)$ ,  $(0, 8/3)$  and  $(16/3, 8]$  respectively. Therefore, the only unique expansions are  $\overline{0}$  and  $\overline{5}$ .

*Proof of Theorem 1.* As  $\mathcal{G}(A) = \mathcal{G}\left(\frac{A-a_0}{a_d-a_0}\right)$ , we have  $\mathcal{G}(\{a_0, a_1, \dots, a_d\}) = \mathcal{G}(\iota \circ r(a_0, a_1, \dots, a_d))$ , with

$$\begin{aligned} r : \Delta_d &\rightarrow \Delta'_d, & (a_0, a_1, \dots, a_d) &\mapsto \left(\frac{a_1 - a_0}{a_d - a_0}, \frac{a_2 - a_0}{a_d - a_0}, \dots, \frac{a_{d-1} - a_0}{a_d - a_0}\right), \\ \iota : \Delta'_d &\rightarrow \mathcal{P}(\mathbb{R}), & (a_1, a_2, \dots, a_{d-1}) &\mapsto \{0, a_1, a_2, \dots, a_{d-1}, 1\}, \end{aligned}$$

and  $\Delta'_d = \{(a_1, a_2, \dots, a_{d-1}) \in \mathbb{R}^{d-1} : 0 < a_1 < a_2 < \dots < a_{d-1} < 1\}$ . As  $r$  is continuous on  $\Delta_d$ , it is sufficient to prove that  $\mathcal{G} \circ \iota$  is continuous on  $\Delta'_d$ .

Let  $\mathbf{a} = (a_1, a_2, \dots, a_{d-1}) \in \Delta'_d$  and  $\varepsilon > 0$  arbitrary but fixed. We will show that  $|\mathcal{G}(\iota(\mathbf{b})) - \mathcal{G}(\iota(\mathbf{a}))| \leq 3\varepsilon$  for all  $\mathbf{b}$  in a neighbourhood of  $\mathbf{a}$ . Let first  $X \subset \Delta'_d$  be a closed neighbourhood of  $\mathbf{a}$  such that  $|q(\iota(\mathbf{b})) - q(\iota(\mathbf{a}))| \leq \varepsilon$  for all  $\mathbf{b} \in X$ . (Note that  $q \circ \iota$  is continuous on  $\Delta'_d$ .) Set

$$\alpha = \min_{\mathbf{b} \in X} q(\iota(\mathbf{b})) - \varepsilon, \quad Y = \{\mathbf{b} \in X : \mathcal{G}(\iota(\mathbf{b})) < \alpha\}.$$

If  $Y = \emptyset$ , then  $X$  is a neighbourhood of  $\mathbf{a}$  with  $|\mathcal{G}(\iota(\mathbf{b})) - \mathcal{G}(\iota(\mathbf{a}))| \leq 2\varepsilon$  for all  $\mathbf{b} \in X$ . Otherwise, let  $\ell \geq 2$  be such that  $\sum_{k=1}^{\ell} \alpha^{-k} \geq (\alpha + \varepsilon - 1)^{-1}$ . Then

$$b_{j+1} - b_j \leq \frac{1}{q(\iota(\mathbf{b})) - 1} \leq \frac{1}{\alpha + \varepsilon - 1} \leq \sum_{k=1}^{\ell} \frac{1}{\alpha^k} \quad (2.3)$$

for all  $(b_1, \dots, b_{d-1}) \in Y$ ,  $0 \leq j < d$ , with  $b_0 = 0$ ,  $b_d = 1$ . Set

$$\delta(\mathbf{a}, \mathbf{b}) = \min_{0 \leq j < d} ((a_{j+1} - a_j) - (b_{j+1} - b_j))$$

(with  $b_0 = a_0 = 0$ ,  $b_d = a_d = 1$ ), and let  $Z \subset X$  be a neighbourhood of  $\mathbf{a}$  such that

$$\frac{a_j}{(\alpha + \varepsilon)^k} - \frac{b_j}{\alpha^k} \leq \delta(\mathbf{a}, \mathbf{b}), \quad \frac{b_j}{(\alpha + \varepsilon)^k} - \frac{a_j}{\alpha^k} \leq \delta(\mathbf{a}, \mathbf{b}) \quad \text{for all } 1 \leq j \leq d, \ 1 \leq k \leq \ell, \quad (2.4)$$

$$\frac{1 - a_j}{(\alpha + \varepsilon)^k} - \frac{1 - b_j}{\alpha^k} \leq \delta(\mathbf{a}, \mathbf{b}), \quad \frac{1 - b_j}{(\alpha + \varepsilon)^k} - \frac{1 - a_j}{\alpha^k} \leq \delta(\mathbf{a}, \mathbf{b}) \quad \text{for all } 0 \leq j < d, \ 1 \leq k \leq \ell, \quad (2.5)$$

for all  $\mathbf{b} = (b_1, \dots, b_{d-1}) \in Z$ . Note that  $\delta(\mathbf{a}, \mathbf{b}) \leq 0$ , thus we also have

$$\frac{a_j}{\alpha + \varepsilon} \leq \frac{b_j}{\alpha}, \quad \frac{1 - a_j}{\alpha + \varepsilon} \leq \frac{1 - b_j}{\alpha}, \quad \frac{b_j}{\alpha + \varepsilon} \leq \frac{a_j}{\alpha}, \quad \frac{1 - b_j}{\alpha + \varepsilon} \leq \frac{1 - a_j}{\alpha} \quad \text{for all } 0 \leq j \leq d. \quad (2.6)$$

For  $\mathbf{b} \in Y \cap Z$  and  $\beta \in (\mathcal{G}(\iota(\mathbf{b})), \alpha]$ , choose  $\mathbf{u} = u_1 u_2 \dots \in \mathcal{U}_{\beta}(\iota(\mathbf{b}))$ . Assume, w.l.o.g., that  $u_1 u_2 \notin \{00, 11\}$ . We show first that  $\mathbf{u}$  does not contain  $\ell$  consecutive zeros or ones. Indeed, suppose that  $u_{i+1} = u_{i+2} = \dots = u_{i+\ell} = 1$  for some  $i \geq 1$ ; then we have

$$\sum_{k=1}^{\infty} \frac{u_{i+k}}{\beta^k} \geq \sum_{k=1}^{\ell} \frac{1}{\beta^k} \geq \sum_{k=1}^{\ell} \frac{1}{\alpha^k} \geq b_{j+1} - b_j$$

for all  $0 \leq j < d$ , hence  $u_i = 1$  because of (2.1); recursively we would obtain that  $u_{i-1} = \dots = u_1 = 1$ , contradicting that  $u_1 u_2 \neq 11$ . Similarly,  $u_{i+1} = u_{i+2} = \dots = u_{i+\ell} = 0$  implies that  $\sum_{k=1}^{\infty} (1 - u_{i+k}) \beta^{-k} \geq b_j - b_{j-1}$  for all  $1 \leq j \leq d$ , hence  $u_i = 0$  because of (2.2), eventually contradicting that  $u_1 u_2 \neq 00$ .

We define the sequence  $\tilde{\mathbf{u}}$  via the relation  $\tilde{u}_{i+k} = a_j$  if  $u_{i+k} = b_j$ . Let now  $i \geq 1$ . We have  $\tilde{u}_{i+k}(\beta + \varepsilon)^{-k} \leq u_{i+k} \beta^{-k}$  for all  $k \geq 1$  because (2.6) implies that  $a_j(\beta + \varepsilon)^{-k} \leq b_j \beta^{-k}$  for all  $0 \leq j \leq d$ . Moreover, (2.4) and (2.6) give that

$$\frac{a_j}{(\beta + \varepsilon)^k} - \frac{b_j}{\beta^k} \leq \frac{a_j}{(\alpha + \varepsilon)^k} - \frac{b_j}{\alpha^k} \leq \delta(\mathbf{a}, \mathbf{b})$$

for all  $1 \leq k \leq \ell$ ,  $1 \leq j \leq d$ . Since  $u_{i+k} \neq 0$  for some  $1 \leq k \leq \ell$ , we have  $\tilde{u}_{i+k}(\beta + \varepsilon)^{-k} \leq u_{i+k} \beta^{-k} + \delta(\mathbf{a}, \mathbf{b})$  for some  $k \geq 1$ . Using (2.1), we get

$$\sum_{k=1}^{\infty} \frac{\tilde{u}_{i+k}}{(\beta + \varepsilon)^k} \leq \sum_{k=1}^{\infty} \frac{u_{i+k}}{\beta^k} + \delta(\mathbf{a}, \mathbf{b}) < b_{j+1} - b_j + \delta(\mathbf{a}, \mathbf{b}) \leq a_{j+1} - a_j \quad \text{when } u_i = b_j \neq 1.$$

Similarly, we obtain from (2.2), (2.5) and (2.6) that

$$\sum_{k=1}^{\infty} \frac{1 - \tilde{u}_{i+k}}{(\beta + \varepsilon)^k} \leq \sum_{k=1}^{\infty} \frac{1 - u_{i+k}}{\beta^k} + \delta(\mathbf{a}, \mathbf{b}) < b_j - b_{j-1} + \delta(\mathbf{a}, \mathbf{b}) \leq a_j - a_{j-1} \quad \text{when } u_i = b_j \neq 0.$$

Therefore, we have  $\tilde{\mathbf{u}} \in \mathcal{U}_{\beta+\varepsilon}(\iota(\mathbf{a}))$ , thus  $\mathcal{G}(\iota(\mathbf{a})) \leq \mathcal{G}(\iota(\mathbf{b})) + \varepsilon$  for all  $\mathbf{b} \in Y \cap Z$ .

For  $\mathbf{b} \in X \setminus Y$ , recall that  $\mathcal{G}(\iota(\mathbf{a})) \leq q(\iota(\mathbf{a})) \leq \alpha + 2\varepsilon \leq \mathcal{G}(\iota(\mathbf{b})) + 2\varepsilon$ . Similarly, we obtain for all  $\mathbf{b} \in Z$  that  $\mathcal{G}(\iota(\mathbf{b})) \leq \mathcal{G}(\iota(\mathbf{a})) + \varepsilon$  when  $\mathbf{a} \in Y$ ,  $\mathcal{G}(\iota(\mathbf{b})) \leq \mathcal{G}(\iota(\mathbf{a})) + 3\varepsilon$  when  $\mathbf{a} \notin Y$ . This gives that  $|\mathcal{G}(\iota(\mathbf{b})) - \mathcal{G}(\iota(\mathbf{a}))| \leq 3\varepsilon$  for all  $\mathbf{b} \in Z$ , thus  $\mathcal{G} \circ \iota$  is continuous at  $\mathbf{a}$ .  $\square$

### 3 Generalised golden ratios over ternary alphabets

#### 3.1 Statements

Komornik, Lai and Pedicini [4] described the function  $m \mapsto \mathcal{G}(m)$  on the interval  $(1, 2]$ . We provide more details for this function, in particular for the set

$$\mathfrak{M} := \{m \in (1, 2] : \mathcal{G}(m) = 1 + \sqrt{m}\}.$$

For  $h \geq 0$ , let  $\tau_h$  be the substitution on the alphabet  $\{0, 1\}$  defined by

$$\tau_h(0) = 0^{h+1}1, \quad \tau_h(1) = 0^h1,$$

and set  $S = \{\tau_h : h \geq 0\}$ . A (right) infinite word  $\mathbf{u}$  is a *limit word* of a sequence of substitutions  $(\sigma_n)_{n \geq 0}$  if there exist words  $\mathbf{u}^{(n)}$  with  $\mathbf{u}^{(0)} = \mathbf{u}$  and  $\mathbf{u}^{(n)} = \sigma_n(\mathbf{u}^{(n+1)})$  for all  $n \geq 0$ . A sequence  $(\sigma_n)_{n \geq 0} \in S^{\mathbb{N}}$  is *primitive* if  $\sigma_n \neq \tau_0$  for infinitely many  $n \geq 0$ . A limit word of a primitive sequence in  $S^{\mathbb{N}}$  starts with  $\sigma_0 \sigma_1 \cdots \sigma_n(0)$  for all  $n \geq 0$  and is therefore unique. If  $\sigma_n = \tau_0$  for all  $n \geq 0$ , then  $1^k 0 \bar{1}$ ,  $k \geq 0$ , and  $\bar{1}$  are limit words of  $(\sigma_n)_{n \geq 0}$ ; we are only interested in  $0\bar{1}$  and  $\bar{1}$ . Therefore, we define the following sets of limit words (or *S-adic words*), where  $S^* = \bigcup_{n \geq 0} S^n$  denotes the set of finite products of substitutions in  $S$ :

$$\begin{aligned} S &= S_{\infty} \cup S_{0\bar{1}} \cup S_{\bar{1}} \quad \text{with} \quad S_{0\bar{1}} = \{\sigma(0\bar{1}) : \sigma \in S^*\}, \quad S_{\bar{1}} = \{\sigma(\bar{1}) : \sigma \in S^*\}, \\ S_{\infty} &= \{\mathbf{u} : \mathbf{u} \text{ is the limit word of a primitive sequence of substitutions in } S^{\mathbb{N}}\}. \end{aligned}$$

*Remark 3.1.* Komornik, Lai and Pedicini [4] observed that the sequences  $\mathbf{u} \in S_{\infty}$  with the leading 0 removed are exactly the standard Sturmian sequences. However, they omitted the word “standard”.

For  $\mathbf{u} = u_0 u_1 \cdots \in \{0, 1\}^{\mathbb{N}}$ , we define  $\mathbf{m}_{\mathbf{u}} \geq 1$  as the unique solution to

$$\mathbf{m}_{\mathbf{u}} = 1 + \sum_{k=0}^{\infty} \frac{u_k}{(1 + \sqrt{\mathbf{m}_{\mathbf{u}}})^k}. \quad (3.1)$$

*Remark 3.2.* We can rewrite (3.1) as

$$1 + \sqrt{\mathbf{m}_{\mathbf{u}}} = 2 + \sum_{k=0}^{\infty} \frac{u_k}{(1 + \sqrt{\mathbf{m}_{\mathbf{u}}})^{k+1}},$$

i.e., Parry’s [6]  $\beta$ -expansion of  $\beta = 1 + \sqrt{\mathbf{m}_{\mathbf{u}}}$  is  $2\mathbf{u}$ . We have  $\mathbf{m}_{\mathbf{u}} = 1$  if and only if  $\mathbf{u} = \bar{0}$ .

For  $\sigma \in S^*$ , we define the interval  $I_{\sigma} = [\mathbf{m}_{\sigma(0\bar{1})}, \mathbf{m}_{\sigma(\bar{1})}] \subset (1, \frac{3+\sqrt{5}}{2}]$ . We define  $\beta_{\sigma} \geq 2$  implicitly via the equation

$$1 + \sum_{k=1}^{\infty} \frac{\tilde{u}_k^{(\sigma)}}{\beta_{\sigma}^k} = (\beta_{\sigma} - 1) \left( 1 + \sum_{k=0}^{\infty} \frac{u_k^{(\sigma)}}{\beta_{\sigma}^{k+1}} \right),$$

where

$$\tilde{u}_0^{(\sigma)} \tilde{u}_1^{(\sigma)} \tilde{u}_2^{(\sigma)} \cdots = \sigma(0\bar{1}), \quad u_0^{(\sigma)} u_1^{(\sigma)} u_2^{(\sigma)} \cdots = \sigma(\bar{1}).$$

Moreover, we let  $\mu_{\sigma}$  denote the coinciding value, i.e.,

$$\mu_{\sigma} := 1 + \sum_{k=1}^{\infty} \frac{\tilde{u}_k^{(\sigma)}}{\beta_{\sigma}^k}.$$

Note that all the numbers and sequences do not change if we replace  $\sigma$  by  $\sigma\tau_0$  since  $\tau_0(0\bar{1}) = 0\bar{1}$  and  $\tau_0(\bar{1}) = \bar{1}$ . Therefore, we can assume that  $\sigma \in S^* \setminus S^*\tau_0$ .

The following propositions recover the main part of Theorem KLP, adding explicit equations giving the generalised golden ratios, which are used for drawing the graph of  $\mathcal{G}$  in Figure 3.1.

**Proposition 3.3.** *The interval  $(1, \frac{3+\sqrt{5}}{2}]$  admits the partition*

$$\{I_\sigma : \sigma \in S^* \setminus S^*\tau_0\} \cup \{\{\mathbf{m}_u\} : u \in \mathcal{S}_\infty\}. \quad (3.2)$$

**Proposition 3.4.** *For  $\sigma \in S^*$  and  $m \in [\mathbf{m}_{\sigma(0\bar{1})}, \mu_\sigma]$ ,  $\mathcal{G}(m)$  is given by*

$$m = 1 + \sum_{k=1}^{\infty} \frac{\tilde{u}_k^{(\sigma)}}{\mathcal{G}(m)^k}.$$

*For  $\sigma \in S^*$  with  $\sigma(1) \neq 1$  and  $m \in [\mu_\sigma, \mathbf{m}_{\sigma(\bar{1})}]$ ,  $\mathcal{G}(m)$  is given by*

$$\frac{m}{\mathcal{G}(m) - 1} = 1 + \sum_{k=1}^{\infty} \frac{u_k^{(\sigma)}}{\mathcal{G}(m)^{k+1}}.$$

**Proposition 3.5.** *We have*

$$\mathfrak{M} = \{\mathbf{m}_u : u \in \mathcal{S} \setminus \{\bar{1}\}\}.$$

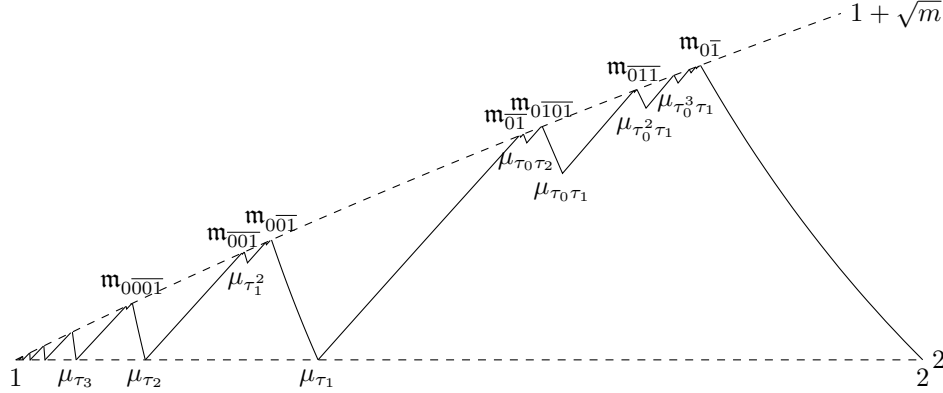


Figure 3.1: A graph of  $\mathcal{G}(m)$ .

### 3.2 Partition of $(1, \frac{3+\sqrt{5}}{2}]$

We first prove Proposition 3.3, using the following lemmas.

**Lemma 3.6.** *Let  $\sigma \in S^*$ . Then  $\sigma$  preserves the lexicographic order on infinite words.*

*Proof.* The lexicographic order on infinite words is preserved by the identity and by  $\sigma \in S$ . By induction on  $n$ , this holds for all  $\sigma \in S^n$ ,  $n \geq 0$ .  $\square$

**Lemma 3.7.** *Let  $\sigma \in S^*$  with  $\sigma(1) \neq 1$ , and write  $\sigma(1) = 0w1$ . Then*

$$\sigma(0\bar{1}) = 0\overline{w0\bar{1}}.$$

*In particular,  $1w0$  is a circular shift of  $\sigma(1)$ .*

*Proof.* Since  $\sigma(1) = \sigma\tau_0(1)$  and  $\sigma(0\bar{1}) = \sigma\tau_0(0\bar{1})$ , we assume w.l.o.g. that  $\sigma = \tau_{h_0}\tau_{h_1} \cdots \tau_{h_n}$  with  $n \geq 0$ ,  $h_n \neq 0$ . Let  $\sigma_k = \tau_{h_k}$ . Then  $\sigma_{[0,n]}(1) = 0w1$  with

$$w = \sigma_0(1)\sigma_{[0,1]}(1) \cdots \sigma_{[0,n-1]}(1)\sigma_{[0,n-1]}(0^{h_{n-1}})\sigma_{[0,n-2]}(0^{h_{n-1}}) \cdots \sigma_0(0^{h_1})0^{h_0}.$$

Let  $v = \sigma_0(1)\sigma_{[0,1]}(1) \cdots \sigma_{[0,n-1]}(1)$ . Then we have  $\sigma_{[0,n]}(0) = 0w01v$  and  $v\sigma_{[0,n]}(1) = w01v$ . Therefore,  $1w0$  is a circular shift of  $\sigma(1)$  and  $\sigma(0\bar{1}) = 0\overline{w0\bar{1}}$ .  $\square$

**Lemma 3.8.** Let  $\mathbf{u} = u_0 u_1 \cdots \in \{0, 1\}^{\mathbb{N}} \setminus \{\bar{0}\}$ . We have  $\mathbf{u} \in \mathcal{S}$  if and only if

$$u_0 u_1 u_2 \cdots \leq u_i u_{i+1} u_{i+2} \cdots \leq 1 u_1 u_2 \cdots \quad \text{for all } i \geq 0. \quad (3.3)$$

*Proof.* Assume that (3.3) holds. Then  $u_0 = 0$  or  $\mathbf{u} = \bar{1} = \tau_0(\bar{1})$ . If  $u_0 = 0$ , let  $h \geq 0$  be minimal such that  $u_{h+1} = 1$ . Then each 1 is followed by  $0^{h+1}1$  or  $0^h 1$ , i.e.,  $\mathbf{u} = \tau_h(\mathbf{u}')$  for some word  $\mathbf{u}' = u'_0 u'_1 \cdots$ . Moreover, we have  $\mathbf{u}' \leq u'_i u'_{i+1} \cdots \leq 1 u'_1 u'_2 \cdots$  for all  $i \geq 0$ . In case  $\mathbf{u}' = \bar{0}$ , we have  $\mathbf{u} = \tau_{h+1}(\bar{1})$ . Therefore, we can repeat the arguments and obtain recursively that  $\mathbf{u}$  is the limit word of a sequence  $(\sigma_n)_{n \geq 0} \in S^{\mathbb{N}}$ . More precisely, we have  $\mathbf{u} \in \mathcal{S}_{0\bar{1}}$  or  $\mathbf{u}$  starts with  $\sigma_{[0,n]}(0)$  for all  $n \geq 0$ , i.e.,  $\mathbf{u} \in \mathcal{S}_{\infty} \cup \mathcal{S}_{\bar{1}}$ .

Consider now  $\mathbf{u} \in \mathcal{S}_{\infty} \cup \mathcal{S}_{0\bar{1}}$ , limit word of  $(\sigma_n)_{n \geq 0} \in S^{\mathbb{N}}$ . Then  $\mathbf{u}$  starts with  $\sigma_{[0,n]}(0)$  for all  $n \geq 0$ . Denote the preimage of  $\mathbf{u}$  by  $\sigma_0$  by  $\mathbf{u}' = u'_0 u'_1 \cdots$ , i.e.,  $\sigma_0(\mathbf{u}') = \mathbf{u}$ . Suppose that  $u_i u_{i+1} \cdots \leq \mathbf{u}$ . Then  $u_i u_{i+1} \cdots$  starts with  $\sigma_0(0)$ , and  $u_i u_{i+1} \cdots = \sigma_0(u'_i u'_{i'+1} \cdots)$  for some  $i' \geq 0$ . This implies that  $u'_i u'_{i'+1} \cdots \leq \mathbf{u}'$ , thus  $u'_i u'_{i'+1} \cdots$  starts with  $\sigma_1(0)$ . Inductively, we obtain that  $u_i u_{i+1} \cdots$  starts with  $\sigma_{[0,n]}(0)$  for all  $n \geq 0$ , i.e.,  $u_i u_{i+1} \cdots = \mathbf{u}$ . Suppose now that  $u_i u_{i+1} \cdots \geq 1 u_1 u_2 \cdots$ . Then  $u_i = 1$  and  $u_{i+1} u_{i+2} \cdots = \sigma_0(u'_i u'_{i'+1} \cdots)$  for some  $i' \geq 0$ , with  $u'_i u'_{i'+1} \cdots \geq 1 u'_1 u'_2 \cdots$ . We get that

$$u_i u_{i+1} \cdots = 1 \sigma_0(1) \sigma_{[0,1]}(1) \sigma_{[0,2]}(1) \cdots = 1 u_1 u_2 \cdots.$$

Therefore, (3.3) holds.

Finally, let  $\mathbf{u} \in \mathcal{S}_{\bar{1}}$ . If  $\mathbf{u} = \bar{1}$ , then (3.3) holds trivially. Otherwise, we have  $\mathbf{u} = \sigma \tau_h(\bar{1})$  with  $\sigma \in S^*$ ,  $h \geq 1$ . Then  $\sigma \tau_{h-1} \tau_j(0\bar{1}) \in \mathcal{S}_{0\bar{1}}$  converges to  $\mathbf{u}$  for  $j \rightarrow \infty$  (in the usual topology of infinite words). By the previous paragraph, (3.3) holds for these words. Hence, it also holds for the limit word  $\mathbf{u}$ .  $\square$

*Proof of Proposition 3.3.* For  $m \in (1, \frac{3+\sqrt{5}}{2}]$ , Parry's  $(1+\sqrt{m})$ -expansions of  $1+\sqrt{m}$  are of the form  $2\mathbf{u} \in \{0, 1, 2\}^{\mathbb{N}}$  with  $\bar{0} < \mathbf{u} \leq \bar{1}$  and are ordered with respect to the lexicographic order. We show that the interval  $(\bar{0}, \bar{1}] \subset \{0, 1, 2\}^{\mathbb{N}}$  admits the partition

$$\{[\sigma(0\bar{1}), \sigma(\bar{1})] : \sigma \in S^* \setminus S^* \tau_0\} \cup \{\{\mathbf{u}\} : \mathbf{u} \in \mathcal{S}_{\infty}\}.$$

Assume that  $\mathbf{u} = u_0 u_1 \cdots \notin \mathcal{S}$ , i.e., (3.3) does not hold. Let  $i \geq 1$  be minimal such that one of the equalities is not satisfied. Suppose first that  $u_i u_{i+1} \cdots < \mathbf{u}$ ; then  $u_{i-1} = 1$ . Similarly to the proof of Lemma 3.8, let  $\sigma_0 \in S$  be such that  $u_0 \cdots u_{i-1} = \sigma_0(u'_0 \cdots u'_{i'-1})$  with  $u'_0 \cdots u'_{i'-1} \neq 0 \cdots 0$ . By minimality of  $i$ , we have  $u'_{i'-1} = 1$ , and  $u'_0 \cdots u'_{i'-1} = 1 \cdots 1$  implies  $i' = 1$ . Therefore, we can define recursively substitutions  $\sigma_j \in S$  until

$$\sigma_{[0,n]}(1) = u_0 u_1 \cdots u_{i-1}.$$

Then we have  $\mathbf{u} < \sigma_{[0,n]}(1) \mathbf{u} < \cdots < \sigma_{[0,n]}(\bar{1})$ . By Lemma 3.6, we have  $\sigma_{[0,n]}(0\bar{1}) < \mathbf{u}$ .

Suppose now that  $u_i u_{i+1} \cdots > 1 u_1 u_2 \cdots$ . Then we have substitutions  $\sigma_k = \tau_{h_k}$  such that

$$\sigma_{[0,n]}(0) = u_0 u_1 \cdots u_{i-1} 1 \sigma_0(1) \sigma_{[0,1]}(1) \cdots \sigma_{[0,n-1]}(1),$$

with  $h_n \neq 0$ . We have  $\mathbf{u} > u_0 \overline{u_1 \cdots u_{i-1} 1}$ , and the latter word is equal to  $\sigma_{[0,n]}(0\bar{1})$  by Lemma 3.7 and its proof. Since  $u_0 \cdots u_{i-1} < \sigma_{[0,n]}(1)$ , we also have  $\mathbf{u} < \sigma_{[0,n]}(\bar{1})$ .

We have seen that each  $\mathbf{u}$  is the limit word of a primitive sequence of substitutions  $\sigma \in S^{\mathbb{N}}$  or between the extremal limit words of a non-primitive sequence  $\sigma \in S^{\mathbb{N}}$ . To see that  $\sigma$  is unique, let  $\mathbf{u}$  and  $\tilde{\mathbf{u}}$  be limit words of two different sequences  $(\sigma_n)_{n \geq 0}$  and  $(\tilde{\sigma}_n)_{n \geq 0}$ . Let  $n \geq 0$  be minimal such that  $\sigma_n \neq \tilde{\sigma}_n$ . Let  $\sigma_n = \tau_h$ ,  $\tilde{\sigma}_n = \tau_j$ , and assume w.l.o.g. that  $h < j$ . Then we have  $\tilde{\mathbf{u}} \leq \tilde{\sigma}_{[0,n]}(\bar{1}) \leq \sigma_{[0,n]}(\bar{0}) < \mathbf{u}$ . Therefore, the intervals are disjoint.  $\square$

### 3.3 Calculating the generalised golden ratio

We now prove that  $\mathcal{G}(m)$  is as in Theorem KLP and Proposition 3.4.

**Lemma 3.9.** Let  $m \in (1, 2]$ ,  $\beta \in [m, m+1]$ , and  $\mathbf{u} = u_0 u_1 \cdots \in \{0, 1\}^{\mathbb{N}} \setminus \{\bar{0}\}$ . Then  $\mathbf{u} \in \mathcal{U}_{\beta}(m)$  if and only if

$$\frac{m}{\beta-1} < 1 + \sum_{k=1}^{\infty} \frac{u_{i+k}}{\beta^k} < m \quad \text{for all } i \geq 0 \text{ such that } u_i = 1.$$



*Proof.* As  $q(\{0, 1, m\}) = 1 + m$ , Theorem P and Remark 2.1 give that  $\mathbf{u} \in \mathcal{U}_\beta(m)$  if and only if  $\sum_{k=0}^{\infty} \frac{u_{i+k}}{\beta^k} \in E_{u_i}$  for all  $i \geq 0$ , with  $E_0 = [0, 1)$  and  $E_1 = (\frac{m}{\beta-1}, m)$ . If  $u_i = 0$ , then we either have  $u_j = 0$  for all  $j \geq i$  and thus  $\sum_{k=0}^{\infty} \frac{u_{i+k}}{\beta^k} = 0 \in E_0$  or there exists  $j > i$  such that  $u_i = \dots = u_{j-1} = 0$ ,  $u_j = 1$ . In the latter case,  $\sum_{k=0}^{\infty} \frac{u_{i+k}}{\beta^k} \in E_1$  implies that  $\sum_{k=0}^{\infty} \frac{u_{i+k}}{\beta^k} \in \beta^{i-j} E_1 \subset E_0$  since  $\beta \geq m$ . This proves the lemma.  $\square$

**Lemma 3.10.** *Let  $\mathbf{u} \in \mathcal{S}_\infty$ ,  $m \in (1, 2]$ . Then we have  $\mathbf{u} \in \mathcal{U}_{1+\sqrt{m}}(m)$  if and only if  $m = \mathbf{m}_\mathbf{u}$ . In particular, we have  $\mathcal{G}(\mathbf{m}_\mathbf{u}) \leq 1 + \sqrt{\mathbf{m}_\mathbf{u}}$ .*

*Proof.* By Lemma 3.9, we have  $\mathbf{u} = u_0 u_1 \dots \in \mathcal{U}_{1+\sqrt{m}}(m)$  if and only if

$$\sqrt{m} < 1 + \sum_{k=1}^{\infty} \frac{u_{i+k}}{(1 + \sqrt{m})^k} < m$$

for all  $i \geq 0$  such that  $u_i = 1$ . By Lemma 3.8 and since  $\mathbf{u}$  is aperiodic,  $u_i = 1$  implies that  $\mathbf{u} < u_{i+1} u_{i+2} \dots < u_1 u_2 \dots$ . Here, the bounds  $\mathbf{u}$  and  $u_1 u_2 \dots$  cannot be improved because, for all  $n \geq 0$ ,  $1\sigma_{[0,n]}(0)$  and  $1\sigma_0(1) \dots \sigma_{[0,n-1]}(1)$  (which is a suffix of  $\sigma_{[0,n]}(0)$ ) are factors of  $\mathbf{u}$ . Therefore, we have  $\mathbf{u} \in \mathcal{U}_{1+\sqrt{m}}(m)$  if and only if

$$\sqrt{m} \leq 1 + \sum_{k=1}^{\infty} \frac{u_k}{(1 + \sqrt{m})^{k+1}} \quad \text{and} \quad 1 + \sum_{k=1}^{\infty} \frac{u_k}{(1 + \sqrt{m})^k} \leq m.$$

This means that  $1 + \sum_{k=1}^{\infty} u_k (1 + \sqrt{m})^{-k} = m$ , i.e.,  $m = \mathbf{m}_\mathbf{u}$ .  $\square$

**Lemma 3.11.** *Let  $\sigma \in S^*$  and  $m > 1$ . There is a unique number  $f_\sigma(m) > 1$  such that*

$$m = 1 + \sum_{k=1}^{\infty} \frac{\tilde{u}_k^{(\sigma)}}{f_\sigma(m)^k}. \quad (3.4)$$

*We have  $f'_\sigma(m) < 0$ ,  $f_\sigma(\mathbf{m}_{\sigma(0\bar{1})}) = 1 + \sqrt{\mathbf{m}_{\sigma(0\bar{1})}}$ ,  $f_\sigma(m) < 1 + \sqrt{m}$  if and only if  $m > \mathbf{m}_{\sigma(0\bar{1})}$ , and  $\sigma(\bar{1}) \notin \mathcal{U}_{f_\sigma(m)}(m)$  if  $m \leq 2$ .*

*Proof.* Let  $h_m(x) = 1 + \sum_{k=1}^{\infty} \tilde{u}_k^{(\sigma)} x^{-k} - m$ . Then  $\lim_{x \rightarrow 1} h_m(x) = \infty$ ,  $\lim_{x \rightarrow \infty} h_m(x) = 1 - m < 0$ ,  $h_m(x)$  is continuous and strictly monotonically decreasing, thus  $f_\sigma(m)$  is the unique solution of  $h_m(x) = 0$ . We have

$$\frac{1}{f'_\sigma(m)} = - \sum_{k=1}^{\infty} \frac{k \tilde{u}_k^{(\sigma)}}{f_\sigma(m)^{k+1}} < 0,$$

in particular  $f_\sigma(m) < f_\sigma(\mathbf{m}_{\sigma(0\bar{1})}) = 1 + \sqrt{\mathbf{m}_{\sigma(0\bar{1})}} < 1 + \sqrt{m}$  for  $m > \mathbf{m}_{\sigma(0\bar{1})}$ .

By Lemma 3.7,  $1\tilde{u}_1^{(\sigma)}\tilde{u}_2^{(\sigma)} \dots$  is a periodic word with the same period as  $\sigma(\bar{1})$ . Therefore, (3.4) and Lemma 3.9 imply that  $\sigma(\bar{1}) \notin \mathcal{U}_{f_\sigma(m)}(m)$ .  $\square$

**Lemma 3.12.** *Let  $\sigma \in S^*$  and  $m > 1$ . There is a unique number  $g_\sigma(m) > 1$  such that*

$$\frac{m}{g_\sigma(m) - 1} = 1 + \sum_{k=0}^{\infty} \frac{u_k^{(\sigma)}}{g_\sigma(m)^{k+1}}. \quad (3.5)$$

*We have  $g'_\sigma(m) > 0$ ,  $g_\sigma(\mathbf{m}_{\sigma(\bar{1})}) = 1 + \sqrt{\mathbf{m}_{\sigma(\bar{1})}}$ ,  $g_\sigma(m) < 1 + \sqrt{m}$  if and only if  $m < \mathbf{m}_{\sigma(\bar{1})}$ , and  $\sigma(\bar{1}) \notin \mathcal{U}_{g_\sigma(m)}(m)$  if  $m \leq 2$ .*

*Proof.* Setting  $h_m(x) = \frac{m}{x-1} - 1 - \sum_{k=0}^{\infty} \frac{u_k^{(\sigma)}}{x^{k+1}}$ , we have

$$h'_m(x) = \sum_{k=0}^{\infty} \frac{(k+1)u_k^{(\sigma)}}{x^{k+2}} - \frac{m}{(x-1)^2} \leq \sum_{k=0}^{\infty} \frac{k+1}{x^{k+2}} - \frac{m}{(x-1)^2} = \frac{1-m}{(x-1)^2} < 0 \quad (3.6)$$

for  $x > 1$ . Similarly to the proof of Lemma 3.11,  $g_\sigma(m)$  is the unique solution of  $h_m(x) = 0$ . (Note that  $g_\sigma(m) = m$  if  $\sigma(\bar{1}) = \bar{1}$ .) We have

$$\frac{1}{g'_\sigma(m)} = (1 - g_\sigma(m)) h'_m(g_\sigma(m)) > 0.$$

Now,  $g_\sigma(m) < 1 + \sqrt{m}$  is equivalent to  $h_m(1 + \sqrt{m}) < 0$ , i.e.,  $1 + \sum_{k=0}^{\infty} u_k^{(\sigma)} (1 + \sqrt{m})^{-k-1} > \sqrt{m}$ . By Remark 3.2, we obtain that  $m < \mathbf{m}_{\sigma(\bar{1})}$ . Since  $1\sigma(\bar{1})$  is a suffix of  $\sigma(\bar{1})$ , (3.5) and Lemma 3.9 give that  $\sigma(\bar{1}) \notin \mathcal{U}_{g_\sigma(m)}(m)$ .  $\square$

**Lemma 3.13.** *Let  $\sigma \in S^*$ . There is a unique  $\mu_\sigma \in (\mathbf{m}_{\sigma(0\bar{1})}, \mathbf{m}_{\sigma(\bar{1})})$  with  $f_\sigma(\mu_\sigma) = g_\sigma(\mu_\sigma)$ . We have  $f_\sigma(\mu_\sigma) = g_\sigma(\mu_\sigma) \geq 2$ , with equality if and only if  $\sigma(\bar{1}) = 0^n \bar{1}$  for some  $n \geq 0$ .*

*If  $m \in [\mathbf{m}_{\sigma(0\bar{1})}, \mu_\sigma]$ , then  $\sigma(\bar{1}) \in \mathcal{U}_\beta(m)$  for all  $\beta > f_\sigma(m)$ , in particular  $\mathcal{G}(m) \leq f_\sigma(m)$ .*

*If  $m \in [\mu_\sigma, \mathbf{m}_{\sigma(\bar{1})}]$ , then  $\sigma(\bar{1}) \in \mathcal{U}_\beta(m)$  for all  $\beta > g_\sigma(m)$ , in particular  $\mathcal{G}(m) \leq g_\sigma(m)$ .*

*Proof.* The number  $\mu_\sigma$  is well defined since  $f'(m) < 0$ ,  $g'(m) > 0$  on  $I_\sigma$ ,

$$f_\sigma(\mathbf{m}_{\sigma(0\bar{1})}) = 1 + \sqrt{\mathbf{m}_{\sigma(0\bar{1})}} > g_\sigma(\mathbf{m}_{\sigma(0\bar{1})}) \text{ and } f_\sigma(\mathbf{m}_{\sigma(\bar{1})}) < 1 + \sqrt{\mathbf{m}_{\sigma(\bar{1})}} = g_\sigma(\mathbf{m}_{\sigma(\bar{1})}).$$

If  $\sigma(\bar{1}) = \bar{1}$ , then  $\mu_\sigma = \beta_\sigma = 2$ . Assume in the following that  $\sigma(\bar{1}) \neq \bar{1}$  and let  $m = 1 + \sum_{k=0}^{\infty} u_k^{(\sigma)} 2^{-k-1}$ , i.e.,  $g_\sigma(m) = 2$ . By Lemma 3.7, we have  $\sigma(\bar{1}) = 0w\bar{1} \leq \bar{w}0\bar{1} = \tilde{u}_1^{(\sigma)} \tilde{u}_2^{(\sigma)} \dots$  for some finite word  $w$ , thus

$$1 + \sum_{k=1}^{\infty} \frac{\tilde{u}_k^{(\sigma)}}{2^k} \geq 1 + \sum_{k=0}^{\infty} \frac{u_k^{(\sigma)}}{2^{k+1}} = m,$$

hence  $f_\sigma(m) \geq 2 = g_\sigma(m)$ . This implies that  $\beta_\sigma \geq 2$ . If  $\beta_\sigma = 2$ , then we must have  $0w = w0$ , i.e.,  $w = 0 \dots 0$ . Therefore,  $\beta_\sigma = 2$  is equivalent to  $\sigma(\bar{1}) = 0^n \bar{1}$  for some  $n \geq 0$ .

Let now  $m \in [\mathbf{m}_{\sigma(0\bar{1})}, \mu_\sigma]$  and  $\beta > f_\sigma(m)$ , or  $m \in [\mu_\sigma, \mathbf{m}_{\sigma(\bar{1})}]$  and  $\beta > g_\sigma(m)$ . Then we also have  $\beta > g_\sigma(m)$  and  $\beta > f_\sigma(m)$  respectively. For  $i \geq 0$  with  $u_i^{(\sigma)} = 1$ , we get

$$\frac{m}{\beta - 1} < 1 + \sum_{k=0}^{\infty} \frac{u_k^{(\sigma)}}{\beta^{k+1}} \leq 1 + \sum_{k=1}^{\infty} \frac{u_{i+k}^{(\sigma)}}{\beta^k} \leq 1 + \sum_{k=1}^{\infty} \frac{\tilde{u}_k^{(\sigma)}}{\beta^k} < m,$$

where the first inequality follows from  $\beta > g_\sigma(m)$  and (3.6), the last inequality from  $\beta > f_\sigma(m)$ , and the middle inequalities are direct consequences of  $\beta \geq 2$  and  $\sigma(\bar{1}) \leq u_{i+1}^{(\sigma)} u_{i+2}^{(\sigma)} \dots \leq \tilde{u}_1^{(\sigma)} \tilde{u}_2^{(\sigma)} \dots$ , which holds by Lemmas 3.8 and 3.7. Thus  $\sigma(\bar{1}) \in \mathcal{U}_\beta(m)$ .  $\square$

The preceding lemmas show that  $\mathcal{G}(m) \leq 1 + \sqrt{m}$  for all  $m \in (1, 2]$ . The next lemma justifies why we have restricted our attention to sequences in  $\{0, 1\}^{\mathbb{N}}$ .

**Lemma 3.14.** *Let  $m \in (1, 2]$ ,  $\beta \leq 1 + \sqrt{m}$ ,  $u_0 u_1 \dots \in \mathcal{U}_\beta(m)$ . Then  $u_i = m$  implies  $u_0 \dots u_i = m \dots m$ .*

*Proof.* If  $u_i = m$ ,  $i \geq 1$ , then (2.2) implies that  $u_{i-1} + \sum_{k=1}^{\infty} \frac{u_{i-1+k}}{\beta^k} > u_{i-1} + \frac{1}{\beta} (1 + \frac{m}{\beta-1}) \geq u_{i-1} + 1$ , thus (2.1) excludes that  $u_{i-1} = 0$  or  $u_{i-1} = 1$ . Recursively, we obtain that  $u_k = m$  for all  $0 \leq k \leq m$ .  $\square$

**Lemma 3.15.** *Let  $m \in (1, 2]$ ,  $\beta < 2$ . Then  $\mathcal{U}_\beta(m)$  is trivial.*

*Proof.* Let  $u_0 u_1 \dots \in \mathcal{U}_\beta(m)$ . By Theorem P, we have  $u_i \neq 1$  for all  $i \geq 0$ . Since  $m \leq 1 + \frac{m}{\beta-1}$ , we have  $m\bar{0} \notin \mathcal{U}_\beta(m)$ , thus Lemma 3.14 implies that  $\mathcal{U}_\beta(m) = \{\bar{0}, \bar{m}\}$ .  $\square$

**Lemma 3.16.** *Let  $m \in (1, 2]$ ,  $\beta \leq 1 + \sqrt{m}$ , and  $u_0 u_1 \dots \in \mathcal{U}_\beta(m) \cap (\{0, 1\}^{\mathbb{N}} \setminus \{\bar{0}\})$ . Then we have  $\inf\{u_{i+1} u_{i+2} \dots : i \geq 0, u_i = 1\} \in \mathcal{S}_\infty \cup \mathcal{S}_{\bar{1}}$ .*

*Proof.* Let  $\tilde{\mathbf{u}} = \tilde{u}_0 \tilde{u}_1 \dots = \inf\{u_{i+1} u_{i+2} \dots : i \geq 0, u_i = 1\}$ . Since  $\tilde{\mathbf{u}} = \bar{1} \in \mathcal{S}_{\bar{1}}$  when  $\tilde{u}_0 = 1$ , we assume in the following that  $\tilde{u}_0 = 0$ . For all  $i \geq 0$  with  $\tilde{u}_i = 1$ , we have

$$\sum_{k=1}^{\infty} \frac{\tilde{u}_{i+k}}{\beta^k} < m - 1 \leq \beta \left( \frac{m}{\beta - 1} - 1 \right) \leq \beta \inf_{i \geq 0: u_i = 1} \sum_{k=1}^{\infty} \frac{u_{i+k}}{\beta^k} \leq \beta \sum_{k=0}^{\infty} \frac{\tilde{u}_k}{\beta^{k+1}} = \sum_{k=1}^{\infty} \frac{\tilde{u}_k}{\beta^k},$$

since  $E_1 = (\frac{m}{\beta-1}, m)$  and  $\beta \leq 1 + \sqrt{m}$ . As  $\beta \geq 2$  by Lemma 3.15, we obtain that  $\tilde{u}_i \tilde{u}_{i+1} \dots < 1 \tilde{u}_1 \tilde{u}_2 \dots$  for all  $i \geq 0$ . By the definition of  $\tilde{\mathbf{u}}$ , we also have  $\tilde{u}_i \tilde{u}_{i+1} \dots \geq \tilde{\mathbf{u}}$ , thus  $\tilde{\mathbf{u}} \in \mathcal{S}$  by Lemma 3.8. Moreover, we have  $\tilde{\mathbf{u}} \notin \mathcal{S}_{0\bar{1}}$  by Lemma 3.7 and the fact that  $\tilde{u}_i \tilde{u}_{i+1} \dots < 1 \tilde{u}_1 \tilde{u}_2 \dots$  for all  $i \geq 0$ .  $\square$

*Remark 3.17.* One obtains similarly that  $\sup\{0u_{i+1}u_{i+2}\cdots : i \geq 0, u_i = 1\} \in \mathcal{S}_\infty \cup \mathcal{S}_{0\bar{1}}$ .

**Proposition 3.18.** *We have*

$$\mathcal{G}(m) = \begin{cases} 1 + \sqrt{m} & \text{if } m \in \{\mathbf{m}_u : u \in \mathcal{S}_\infty\}, \\ f_\sigma(m) & \text{if } m \in [\mathbf{m}_{\sigma(0\bar{1})}, \mu_\sigma], \sigma \in S^*, \\ g_\sigma(m) & \text{if } m \in [\mu_\sigma, \mathbf{m}_{\sigma(\bar{1})}], \sigma \in S^*. \end{cases}$$

*Proof.* Let  $\beta \geq 2$ ,  $u \in \mathcal{U}_\beta(m) \cap \{0, 1\}^\mathbb{N}$  and  $\tilde{u}$  as in Lemma 3.16. Then  $\tilde{u} \in \mathcal{U}_{\tilde{\beta}}(m)$  for all  $\tilde{\beta} > \beta$ . If  $\tilde{u} \in \mathcal{S}_\infty$ , then Lemma 3.10 gives that  $\beta \geq 1 + \sqrt{m}$ . If  $\tilde{u} = \sigma(\bar{1})$ ,  $\sigma \in S^*$ , then  $\beta \geq \max\{f_\sigma(m), g_\sigma(m)\}$  by Lemmas 3.11 and 3.12. This implies that  $\beta \geq f_\sigma(m)$  if  $m \in [\mathbf{m}_{\sigma(0\bar{1})}, \mu_\sigma]$ ,  $\beta \geq g_\sigma(m)$  if  $m \in [\mu_\sigma, \mathbf{m}_{\sigma(\bar{1})}]$ , and  $\beta \geq 1 + \sqrt{m}$  otherwise. The opposite inequalities are also proved in Lemmas 3.10, 3.11 and 3.12.  $\square$

The previous lemmas prove Propositions 3.4 and 3.5 and the main part of Theorem KLP.

*Proof of Theorem 2.* By Lemmas 3.11 and 3.12,  $\mathcal{G}(m)$  is differentiable on  $(1, 2] \setminus (\mathfrak{M} \cup \{\mu_\sigma : \sigma \in S^*\})$ . By Propositions 3.3 and 3.18, Lemmas 3.11 and 3.12, and the continuity of  $\mathcal{G}$  on  $(1, 2]$ , the total variation is

$$\sum_{\sigma \in S^* \setminus S^* \tau_0} (\mathcal{G}(\mathbf{m}_{\sigma(0\bar{1})}) - \mathcal{G}(\mu_\sigma)) + \sum_{\sigma \in S^* \setminus S^* \tau_0 : \sigma(1) \neq 1} (\mathcal{G}(\mathbf{m}_{\sigma(\bar{1})}) - \mathcal{G}(\mu_\sigma)).$$

As  $\lim_{m \rightarrow 1^+} \mathcal{G}(m) = 2 = \mathcal{G}(2)$ , the two sums are equal. For  $m \in (\mu_\sigma, \mathbf{m}_{\sigma(\bar{1})})$ ,  $\sigma \in S^*$ ,  $\sigma(1) \neq 1$ , we have

$$\begin{aligned} \frac{1}{\mathcal{G}'(m)} &= \frac{m}{\mathcal{G}(m) - 1} - (\mathcal{G}(m) - 1) \sum_{k=1}^{\infty} \frac{(k+1)u_k^{(\sigma)}}{\mathcal{G}(m)^{k+2}} = 1 - \sum_{k=1}^{\infty} \left(k - \frac{k+1}{\mathcal{G}(m)}\right) \frac{u_k^{(\sigma)}}{\mathcal{G}(m)^{k+1}} \\ &> 1 - \sum_{k=1}^{\infty} \left(k - \frac{k+1}{\mathcal{G}(m)}\right) \frac{1}{\mathcal{G}(m)^{k+1}} = 1 - \frac{1}{\mathcal{G}(m)^2} \geq \frac{3}{4}, \end{aligned}$$

using that  $k - \frac{k+1}{\mathcal{G}(m)} \geq 0$  for all  $k \geq 1$ . Therefore, we have

$$\sum_{\sigma \in S^* \setminus S^* \tau_0 : \sigma(1) \neq 1} (\mathcal{G}(\mathbf{m}_{\sigma(\bar{1})}) - \mathcal{G}(\mu_\sigma)) < \frac{4}{3} \sum_{\sigma \in S^* \setminus S^* \tau_0 : \sigma(1) \neq 1} (\mathbf{m}_{\sigma(\bar{1})} - \mu_\sigma) = \frac{4}{3} \left(1 - \sum_{\sigma \in S^* \setminus S^* \tau_0} (\mu_\sigma - \mathbf{m}_{\sigma(0\bar{1})})\right).$$

We have  $\beta_{\tau_h} = 2$  for all  $h \geq 0$  since  $\tau_h(\bar{1}) = \overline{0^h 1}$  and  $\tau_h(0\bar{1}) = \overline{0 0^h 1}$ , thus  $\mu_{\tau_h} = 2^{h+1}/(2^{h+1} - 1)$ , and  $\beta = 1 + \sqrt{\mathbf{m}_{\tau_h(0\bar{1})}}$  satisfies  $\beta^{h+3} - 2\beta^{h+2} - 1 = \beta^2 - 2\beta$ , hence  $\mu_{\tau_0} - \mathbf{m}_{\tau_0(0\bar{1})} \approx 0.24512$  (and  $\mu_\sigma = \mu_{\tau_0}$  when  $\sigma$  is the identity,  $\mathbf{m}_{0\bar{1}} = \mathbf{m}_{\tau_0(0\bar{1})}$ ),  $\mu_{\tau_1} - \mathbf{m}_{\tau_1(0\bar{1})} \approx 0.05136$ . This gives that  $\sum_{\sigma \in S^* \setminus S^* \tau_0} (\mu_\sigma - \mathbf{m}_{\sigma(0\bar{1})}) > 1/4$ , thus the total variation is less than 2.

The derivative is unbounded because we have, for all  $m \in (\mathbf{m}_{\sigma(0\bar{1})}, \mu_\sigma)$  with  $\sigma \in \tau_h S^*$ ,

$$\left| \frac{1}{\mathcal{G}'(m)} \right| = \sum_{k=1}^{\infty} \frac{k \tilde{u}_k^{(\sigma)}}{\mathcal{G}(m)^{k+1}} \leq \sum_{k=h+1}^{\infty} \frac{k}{2^{k+1}} = \frac{h+2}{2^{h+1}}. \quad \square$$

*Proof of Theorem 4.* Note that the map  $\iota : m \mapsto \frac{m}{m-1}$  is an order-reversing involution on  $(1, \infty)$ . By Proposition 3.18, we have  $m = \iota(\mathcal{G}(m)^h)$  and thus  $\mathcal{G}(m) = \iota(m)^{1/h}$  for all  $m \in [\mathbf{m}_{\tau_{h-1}(0\bar{1})}, \mu_{\tau_{h-1}}]$ ,  $h \geq 1$ . Moreover,  $\iota(m) \geq \mathbf{m}_{\tau_{h-1}(0\bar{1})}$  is equivalent to  $1 + \sqrt{\iota(m)} \geq m^{1/h}$  by Lemma 3.11, and  $\mu_{\tau_{h-1}} = 2$  by the proof of Theorem 2. For  $2^h \leq m \leq (1 + \sqrt{\iota(m)})^h$ , we have thus  $\mathcal{G}(\iota(m)) = m^{1/h}$ .  $\square$

### 3.4 Hausdorff dimension of $\mathfrak{M}$

In this section we show that the Hausdorff dimension of  $\mathfrak{M}$  is 0.

*Proof of Theorem 3.* It suffices to show that  $\dim_H(\mathcal{G}(\mathfrak{M})) = 0$  because  $\mathcal{G} : \mathfrak{M} \rightarrow \mathbb{R}$  is given by  $\mathcal{G}(m) = 1 + \sqrt{m}$ , and  $1 + \sqrt{m}$  is bi-Lipschitz on the interval  $(1, 2]$ .

Given  $m \in \mathfrak{M}$ , we know by Proposition 3.5 that  $m = \mathfrak{m}_{\mathbf{u}}$  for some  $\mathbf{u} \in \mathcal{S} \setminus \{\bar{1}\}$ . Remark 3.2 states that  $2\mathbf{u}$  is also the  $\beta$ -expansion of  $\beta = 1 + \sqrt{\mathfrak{m}_{\mathbf{u}}}$ . Therefore, for each  $n \in \mathbb{N}$  we have

$$1 + \sqrt{\mathfrak{m}_{\mathbf{u}}} \in C_{2u_1 \dots u_n} := \{\beta > 1 : \text{the } \beta\text{-expansion of } \beta \text{ starts with } 2u_1 \dots u_n\}.$$

We have  $C_{2u_1 \dots u_n} \subset [2, \infty)$  and, hence, the diameter of  $C_{2u_1 \dots u_n}$  is at most  $2^{-n}$ , e.g., by a lemma of Schmeling [10, Lemma 4.1].

We now prove  $\dim_H(\mathcal{G}(\mathfrak{M})) = 0$  by explicitly constructing a cover. We introduce the set

$$L_n := \{u_1 \dots u_n \in \{0, 1\}^n : u_1 \dots u_n \text{ is a prefix of an element of } \mathcal{S}\}.$$

For each  $n \in \mathbb{N}$  we have

$$\mathcal{G}(\mathfrak{M}) \subset \bigcup_{u_1 \dots u_n \in L_n} C_{2u_1 \dots u_n}.$$

So the set  $\{C_{2u_1 \dots u_n} : u_1 \dots u_n \in L_n\}$  is a cover of  $\mathcal{G}(\mathfrak{M})$ . Let  $s > 0$  be arbitrary and  $\mathcal{H}^s(\cdot)$  denote the  $s$ -dimensional Hausdorff measure. We observe

$$\mathcal{H}^s(\mathcal{G}(\mathfrak{M})) \leq \lim_{n \rightarrow \infty} \sum_{u_1 \dots u_n \in L_n} \text{Diam}(C_{2u_1 \dots u_n})^s \leq \lim_{n \rightarrow \infty} \frac{\#L_n}{2^{ns}}.$$

As was pointed out in Remark 3.1, every element of  $\mathcal{S}$  is a Sturmian sequence. Thus it is a consequence of [5, Theorem 2.2.36] that  $\#L_n$  grows at most polynomially in  $n$ . Therefore  $\lim_{n \rightarrow \infty} \#L_n 2^{-ns} = 0$  and  $\dim_H(\mathfrak{M}) \leq s$ . Since  $s$  is arbitrary we are done.  $\square$

## 4 Behaviour at the generalised golden ratio

In this section we discuss the behaviour of the univoque set at the generalised golden ratio. It was observed in [1] that when  $\beta = \mathcal{G}(\{0, 1, \dots, m\})$  for some positive integer  $m$ , then every  $x \in (0, \frac{m}{\beta-1})$  either has a countable infinite of expansions, or a continuum of expansions. In other words  $\mathcal{U}_{\mathcal{G}(\{0, 1, \dots, m\})}(\{0, 1, \dots, m\})$  is still trivial. However, Lemma 3.10 demonstrates that this is not always the case. Indeed the following result is an immediate consequence of this lemma.

**Proposition 4.1.** *If  $\mathbf{u} \in \mathcal{S}_{\infty}$  then  $\mathcal{U}_{\mathcal{G}(\mathbf{m}_{\mathbf{u}})}(\mathbf{m}_{\mathbf{u}})$  is non-trivial.*

In [9] it was shown that the smallest  $\beta \in (1, 2)$  for which an  $x$  has precisely two expansions over the alphabet  $\{0, 1\}$  was  $\beta_2 \approx 1.71064$ . As such, there is a small gap between the golden ratio for the alphabet  $\{0, 1\}$ , and the smallest  $\beta$  for which an  $x$  has precisely two expansion. As we show below, for certain alphabets it is possible that an  $x$  has precisely two expansions at the golden ratio.

**Proposition 4.2.** *For every  $\mathbf{u} \in \mathcal{S}_{\infty}$ , the number  $\mathbf{m}_{\mathbf{u}}/\mathcal{G}(\mathbf{m}_{\mathbf{u}})$  has precisely two expansions in base  $\mathcal{G}(\mathbf{m}_{\mathbf{u}})$  over the alphabet  $\{0, 1, \mathbf{m}_{\mathbf{u}}\}$ .*

*Proof.* Let  $\beta = \mathcal{G}(\mathbf{m}_{\mathbf{u}}) = 1 + \sqrt{\mathfrak{m}_{\mathbf{u}}}$  and let  $\mathbf{m}_{\mathbf{u}}/\beta = \sum_{k=1}^{\infty} v_k \beta^{-k}$  be an expansion of  $\mathbf{m}_{\mathbf{u}}/\beta$  over the alphabet  $\{0, 1, \mathbf{m}_{\mathbf{u}}\}$ . Since  $\mathbf{m}_{\mathbf{u}} > \frac{\mathbf{m}_{\mathbf{u}}}{\beta-1}$ , we have  $v_1 \in \{1, \mathbf{m}_{\mathbf{u}}\}$ , thus  $\sum_{k=1}^{\infty} v_{k+1} \beta^{-k}$  equals  $\mathbf{m}_{\mathbf{u}} - 1$  and 0 respectively. Clearly, 0 has a unique expansion, and  $\mathbf{m}_{\mathbf{u}} - 1$  has the expansion  $u_1 u_2 \dots$  by (3.1), which is also unique by Lemma 3.10.  $\square$

Proposition 4.1 and Proposition 4.2 imply Theorem 5.

**Acknowledgements** The authors are grateful to Vilmos Komornik for posing the questions that lead to this research.

## References

- [1] S. Baker, *Generalized golden ratios over integer alphabets*, Integers **14** (2014), Paper No. A15, 28 pp.
- [2] Z. Daróczy, I. Katai, *Univoque sequences*, Publ. Math. Debrecen **42** (1993), 397–407.

- [3] P. Erdős, I. Joó, V. Komornik, *Characterization of the unique expansions  $1 = \sum_{i=1}^{\infty} q^{-n_i}$  and related problems*, Bull. Soc. Math. Fr. **118** (1990), 377–390.
- [4] V. Komornik, A.C. Lai, M. Pedicini, *Generalized golden ratios of ternary alphabets*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 4, 1113–1146.
- [5] M. Lothaire, *Algebraic combinatorics on words*, Encyclopedia of Mathematics and its Applications, 90. Cambridge University Press, Cambridge, 2002. xiv+504 pp. ISBN: 0-521-81220-8
- [6] W. Parry, *On the  $\beta$ -expansions of real numbers*, Acta Math. Acad. Sci. Hung. **11** (1960) 401–416.
- [7] M. Pedicini, *Greedy expansions and sets with deleted digits*, Theoret. Comput. Sci. **332** (2005), no. 1–3, 313–336.
- [8] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hung. **8** (1957) 477–493.
- [9] N. Sidorov, *Expansions in non-integer bases: lower, middle and top orders*, J. Number Th. **129** (2009), 741–754.
- [10] J. Schmeling, *Symbolic dynamics for  $\beta$ -shifts and self-normal numbers*, Ergodic Theory Dynam. Systems **17** (1997), no. 3, 675–694.